Geometry of the dot product

Recall, given $v=\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $w=\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, $v \cdot w = v_1w_1 + \cdots + v_nw_n = v^T w$

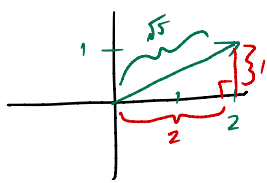
Properties:

- $v \cdot w = w \cdot v$
- $c(v \cdot w) = (cv) \cdot w = v \cdot (cw)$
- $v \cdot (u+w) = v \cdot u + v \cdot w$

} Follows from matrix multiplication properties

Def: The length of a vector $v=\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is $\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

Ex: $v=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has length $\|v\| = \sqrt{1^2+2^2} = \sqrt{5}$



Properties: Let $v \in \mathbb{R}^n$, $c \in \mathbb{R}$

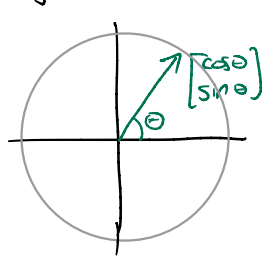
- $\|v\| \geq 0$
- $\|v\| = 0 \iff v = 0$
- $\|cv\| = |c| \, \|v\|$

Def: A unit vector is a vector of length 1

Ex: Unit vectors in $\mathbb{R}^2$ are of the form $v=\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

These are the vectors with endpoints on the unit circle (points with distance one away from the origin)

Indeed, $\|v\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

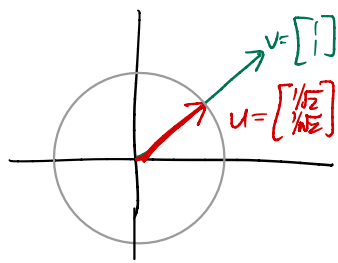


Let $v \in \mathbb{R}^n$, $v \neq 0$. Then $u = \frac{1}{\|v\|} \cdot v$ is the unit vector pointing in the direction of v:

- Since $u = \frac{1}{\|v\|} \cdot v$, u is a positive scalar multiple of v
- $\|u\| = \sqrt{u \cdot u} = \sqrt{\left(\frac{1}{\|v\|}v\right) \cdot \left(\frac{1}{\|v\|}v\right)} = \sqrt{\frac{1}{\|v\|^2} v \cdot v} = \frac{1}{\|v\|}\sqrt{v \cdot v} = \frac{\|v\|}{\|v\|} = 1$

Ex: Let $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\|v\| = \sqrt{2}$

and so $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$



Angles

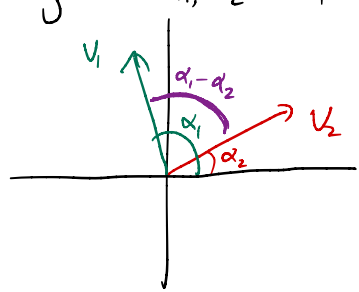
Let v_1 and v_2 be unit vectors in $\mathbb{R}^2$

Then $v_1 = \begin{bmatrix} \cos\alpha_1 \\ \sin\alpha_1 \end{bmatrix}$, $v_2 = \begin{bmatrix} \cos\alpha_2 \\ \sin\alpha_2 \end{bmatrix}$ for some angles $\alpha_1, \alpha_2 \in [0, 2\pi)$

$\Rightarrow$ $v_1 \cdot v_2 = \cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2$

$= \cos(\alpha_1 - \alpha_2)$ ← trig identity

Angle between u and v



So, if u and v are unit vectors, then $u \cdot v = \cos\theta$, where θ is the angle between u and v

Now suppose v_1 and v_2 are not necessarily unit vectors

Then $u_1 = \frac{v_1}{\|v_1\|}$ and $u_2 = \frac{v_2}{\|v_2\|}$ are unit vectors.

Let θ be the angle between u_1 and u_2 (and thus between v_1 and v_2). Then:

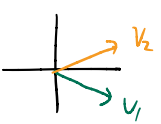
$$\cos\theta = u_1 \cdot u_2 = \left(\frac{1}{\|v_1\|} v_1\right) \cdot \left(\frac{1}{\|v_2\|} v_2\right) = \frac{1}{\|v_1\|\|v_2\|} v_1 \cdot v_2$$

So $v_1 \cdot v_2 = \|v_1\|\|v_2\| \cos\theta$, where θ is the angle between v_1 and v_2

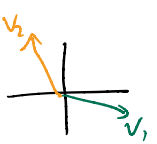
Although we only thought about $\mathbb{R}^2$, this is true for any $v_1, v_2 \in \mathbb{R}^n$. (since any two vectors live in a plane)

This tells us about the relative directions in which v_1 and v_2 point. Suppose $v_1, v_2 \neq \vec{0}$.

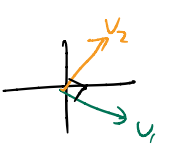
- $v_1 \cdot v_2 > 0 \iff \cos\theta > 0 \iff -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



- $v_1 \cdot v_2 < 0 \iff \cos\theta < 0 \iff \frac{\pi}{2} < \theta < \frac{3\pi}{2}$



- $v_1 \cdot v_2 = 0 \iff \cos\theta = 0 \iff v_1 \perp v_2$.



Two vectors $v, w \in \mathbb{R}^n$ are called orthogonal if $v \cdot w = 0$.

Orthogonal Subspaces

Def: Two subspaces V and W of $\mathbb{R}^n$ are orthogonal if $v \cdot w = 0$ for all $v \in V$ and $w \in W$. Write $V \perp W$.

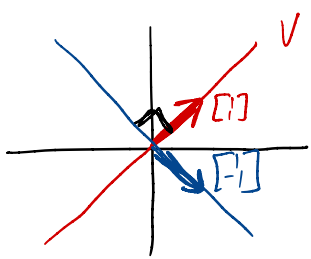
Ex: Show $V = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ and $W = \text{Span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ are orthogonal.

Let $v \in V$ and $w \in W$.

Then $v = x\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}x\\x\end{bmatrix}$ and $w = y\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}y\\-y\end{bmatrix}$ for some $x, y \in \mathbb{R}$

Since $v \cdot w = \begin{bmatrix}x\\x\end{bmatrix} \cdot \begin{bmatrix}y\\-y\end{bmatrix} = xy - xy = 0$, we have that the subspaces are orthogonal.

Geometrically



V and W are perpendicular lines.

Ex: The planes $V = \left\{\begin{bmatrix}x\\y\\0\end{bmatrix} \mid x, y \in \mathbb{R}\right\}$ and $W = \left\{\begin{bmatrix}0\\y\\z\end{bmatrix} \mid y, z \in \mathbb{R}\right\}$ are not orthogonal.

Since $\begin{bmatrix}1\\0\\0\end{bmatrix} \in V$ and $\begin{bmatrix}0\\1\\1\end{bmatrix} \in W$

but $\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\1\end{bmatrix} = 1 \neq 0$

(even though the planes look perpendicular)